Cournot competition

IE 5545: Decision Analysis
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1 Competition between firms

In these notes, we illustrate the concept of Nash equilibrium by applying it to study quantity competition between firms. In such a competition, firms compete by choosing the quantity of goods they each bring to the market, and where the price of the good depends on the total quantity of goods in the market.

1.1 Quantity competition

We begin by describing a model of oligopoly competition first studied by Cournot (1838). The model analyzes a firm's decision problem determining the production quantity of a commodity, when it faces competition from other firms in the market. For simplicity, we will focus on the case with two firms, but the analysis can be easily extended to the case of N firms.

Consider an oligopoly with two firms $i \in \{1, 2\}$, who each produce a single homogeneous commodity. Each firm i must decide its production level $q_i \geq 0$. The cost of production for firm i is given by $c_i(q_i)$. Once the production levels are determined (and produced), the total quantity of the commodity $Q = q_1 + q_2$ will be sold in the market at the "market clearing" price $P(Q)$, where the market is said to clear when the total supply of goods is equal to the total demand at that price. (Hence $P(\cdot)$ is also called the *inverse demand function*.) Each firm seeks to maximize its profit, which is given by its revenue minus the cost of production:

$$\begin{aligned} \pi_i(q_i, q_{-i}) &= q_i P(Q) - c_i(q_i) \\ &= q_i P(q_i + q_{-i}) - c_i(q_i). \end{aligned}$$

We assume the functions $P(\cdot)$ and $\{c_i(\cdot) : i \in \{1, 2\}\}$ are commonly known to both the firms. Thus, we obtain a strategic form game with complete information.

A (pure) Nash equilibrium for the oligopoly competition is a pair of production levels (q_1^*, q_2^*) such that, fixing firm 2's production at q_2^*, firm 1 maximizes her profit by producing q_1^*, and vice versa. Formally, we have (q_1^*, q_2^*) is a pure NE if and only if

$$\begin{aligned} &q_1^* \text{ maximizes } \pi_1(q_1, q_2^*), \quad \text{over } q_1 \geq 0; \text{ and,} \\ &q_2^* \text{ maximizes } \pi_2(q_1^*, q_2), \quad \text{over } q_2 \geq 0. \end{aligned}$$

We want to know if there exists a Nash equilibrium for this model, and if so, what are the production levels of the two firms in that equilibrium. To make things little simpler, we make some further assumptions on the model:

1. The inverse demand function $P(Q)$ is given by

$$P(Q) = (a - bQ)^+ = \max\{a - Q, 0\}.$$

   In other words, the price is given by $a - bQ$ if $Q \leq \frac{a}{b}$, and 0 otherwise.

   This assumption models the scenario where if the total quantity Q goes beyond a level $\frac{a}{b}$, the price in the market drops to a very low level. Furthermore, a denotes the maximum price that can be sustained in the market.

2. The cost of production for the two firms is given by

$$c_i(q_i) = cq_i.$$

   This means that the cost of production is linear with no fixed costs, and the marginal cost of production is same for the two firms, and is given by c. (The assumption that the marginal cost of production is same for the two firms models the scenario where the two firms are using the same "technology" to produce the commodity.)

3. The parameters satisfy $a > c$. If this is not true, i.e., if $a \leq c$, then the marginal cost of production c is at least than the maximum price in the market, implying that the firms would be better off not producing anything.

Given these assumption, we want to know whether a Nash equilibrium exists, and if so, the quantities each firm produces in equilibrium.

1.2 Best response sets

To find the equilibrium, we first start by writing down the best response map for each firm. Because of symmetry, it is enough to compute the best response map $BR_1(q_2)$ for player 1 for any choice of quantity produced by firm 2.

Now, suppose that $q_2 \geq \frac{a}{b}$. Then, $Q = q_1 + q_2 \geq q_2 \geq \frac{a}{b}$ and hence $P(Q) = \max\{a - bQ, 0\} = 0$. Thus, for any positive choice of q_1, the firm gets zero revenue and incurs a positive production cost. This yields strictly less profit that producing nothing. Thus, we conclude that $BR_1(q_2) = \{0\}$ if $q_2 \geq \frac{a}{b}$.

Next, suppose $q_2 \in [0, \frac{a}{b})$. If firm 1 produces any quantity $q_1 > \frac{a}{b} - q_2$, then again the firm gets zero revenue and incurs a positive cost. This is strictly worse than producing nothing. Thus, we obtain that no such q_1 can belong to $BR_1(q_2)$. Thus, if $q_1 \in BR_1(q_2)$ then $q_1 \in [0, \frac{a}{b} - q_2]$.

For any such $q_1 \in [0, \frac{a}{b} - q_2]$, we have $Q = q_1 + q_2 < \frac{a}{b}$, and hence $P(Q) = \max\{a - bQ, 0\} = a - bQ$. From this, we can write the payoff function of firm 1 as

$$\begin{aligned}
\pi_1(q_1, q_2) &= q_1 P(Q) - cq_1 \\
&= q_1(a - bQ) - cq_1 \\
&= q_1(a - b(q_1 + q_2)) - cq_1 \\
&= q_1(a - c - bq_2) - bq_1^2.
\end{aligned}$$

We seek the global maximizer of this function of q_1 (for fixed value of q_2) over the set $[0, \frac{a}{b} - q_2]$. Given that $\pi_1(q_1, q_2)$ is a differentiable function of q_1, the global maximizer must be at a stationary point (i.e., a point where the derivative is zero) or at the boundary. The boundary points are (1) $q_1 = 0$ with $\pi_1(q_1, q_2) = \pi_1(0, q_2) = 0$, and (2) $q_1 = \frac{a}{b} - q_2$, with $\pi_1(q_1, q_2) = -cq_1 < 0$. Thus, the global maximizer cannot be at $q_1 = \frac{a}{b} - q_2$, and must be either at $q_1 = 0$ or at any stationary point.

Computing the derivative of π_1 with respect to q_1, we obtain

$$\left.\frac{\partial \pi_1(q_1, q_2)}{\partial q_1}\right|_{q_1 = q_1^*} = a - c - bq_2 - 2bq_1.$$

Now, if $a - c - bq_2 < 0$, then the derivative is always negative, and there is no stationary point. Thus, the global maximizer is $q_1^* = 0$ if $a - c - bq_2 < 0$, i.e, if $q_2 > \frac{a-c}{b}$.

On the other hand, if $a - c - bq_2 \geq 0$, then the stationary point is one where $a - c - bq_2 - 2bq_1 = 0$, which yields $q_1 = \frac{a-c-bq_2}{2b} \in [0, \frac{a}{b} - q_2]$. The payoff of firm 1 at this value of q_1 is given by

$$\begin{aligned}
\pi(q_1, q_2) &= q_1(a - c - bq_2) - bq_1^2 \\
&= \left(\frac{a - c - bq_2}{2b}\right)(a - c - bq_2) - b\left(\frac{a - c - bq_2}{2b}\right)^2 \\
&= \frac{(a - c - bq_2)^2}{4b} \\
&> 0.
\end{aligned}$$

Because the payoff is positive (i.e., more than the payoff at $q_1 = 0$), we conclude that the global maximizer is $q_1^* = \frac{a-c-bq_2}{2b}$ if $a - c - bq_2 \geq 0$, i.e., if $q_2 \leq \frac{a-c}{b}$.

Summarizing, we obtain the best response set $BR_1(q_2)$ as follows:

$$BR_1(q_2) = \begin{cases} \{0\} & \text{if } q_2 > \frac{a-c}{b}; \\ \left\{\frac{a-c-bq_2}{2b}\right\} & \text{if } q_2 \leq \frac{a-c}{b}. \end{cases} \tag{1}$$

By symmetry, we obtain

$$BR_2(q_1) = \begin{cases} \{0\} & \text{if } q_1 > \frac{a-c}{b}; \\ \left\{\frac{a-c-bq_1}{2b}\right\} & \text{if } q_1 \leq \frac{a-c}{b}. \end{cases} \tag{2}$$

1.3 Existence, uniqueness and computation of equilibrium

In an NE (q_1^*, q_2^*), the quantity q_i^* maximizes $\pi_i(q_i, q_{-i}^*)$ over all q_i. In other words, we require $q_1^* \in BR_1(q_2^*)$ and $q_2^* \in BR_2(q_1^*)$.

First, observe that in any NE (q_1^*, q_2^*), we must have $q_1^* > 0$ and $q_2^* > 0$. To see this, suppose, for the sake of contradiction, $q_1^* = 0$. Since $q_2^* \in BR_2(q_1^*) = BR_2(0)$, we conclude from (2) that $q_2^* = \frac{a-c-b(0)}{2b} = \frac{a-c}{2b}$. But, then from (1), we have $BR_1(q_2^*) = \{\frac{a-c-bq_2^*}{2b}\} = \{\frac{a-c}{4b}\}$, and hence $q_1^* \notin BR_1(q_2^*)$, contradicting that (q_1^*, q_2^*) is an NE. We arrive at a similar contradiction if we suppose $q_2^* = 0$.

Thus, we conclude that $q_1^* > 0$ and $q_2^* > 0$ in any NE. By (1) and (2), we conclude that q_1^* and q_2^* must satisfy

$$\begin{aligned}
q_1^* &= \frac{a - c - bq_2^*}{2b} \\
q_2^* &= \frac{a - c - bq_1^*}{2b}.
\end{aligned}$$

Solving this set of equations, we obtain the following unique solution:

$$q_1^* = q_2^* = \frac{a-c}{3b} \qquad\qquad Q^* = q_1^* + q_2^* = \frac{2(a-c)}{3b}.$$

Thus, in this model, we obtain a unique pure NE. This equilibrium is also referred to as a Cournot equilibrium. Observe that the price in this equilibrium is given by $P(Q^*) = a - Q = \frac{1}{3}a + \frac{2}{3}c$, which is strictly greater than the cost of production c. Hence both firms make positive profit in the Cournot equilibrium. (This is in contrast to a setting with price competition, where both firms set prices to undercut each other until they end up selling their goods at the cost of production and make zero profit. See Bertrand competition for more information.)

1.4 Comparison with monopolist/collusion

Now that we explicitly have the Cournot equilibrium, we can compare its properties with a setting where there is a single firm, a monopolist, or equivalently, where the two firms collude to set quantities. Such a setting can be modeled by assuming that the two firms are operated by a single monopolist, who is interested in maximizing the total profit $TP(q_1, q_2) = \pi_1(q_1, q_2) + \pi_2(q_1, q_2)$. By similar arguments as before, it is straightforward to obtain that a monopolist would produce a total quantity Q_m given by

$$Q_m = \frac{a-c}{2b}.$$

The price $P(Q_m)$ under collusion is then given by

$$P(Q_m) = a - bQ_m = \frac{1}{2}a + \frac{2}{2}c.$$

Observe first that $Q_m < Q^*$ and $P(Q_m) > P(Q^*)$. Thus, in oligopoly quantity competition, we infer from the analysis that there is overproduction in the market. Equivalently, under collusion, the supply of the good is restricted by the firms so that they can command a higher price.

