Pure Nash Equilibrium

IE 5545: Decision Analysis
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Until now, we were asking which strategies a rational player will avoid playing in a game. This led us to the notion of *dominance* (strict and weak), from which we developed the notion of IESDS, i.e. iterated elimination of strictly dominated strategies. When a strategic form game leads to a unique strategy profile at the end of the IESDS process, we took that strategy profile to the solution to the game.

However, it is easy to see that the IESDS process does not always lead to a solution. In games like matching pennies, the IESDS process does not eliminate any strategies at all. Thus, we need some other method to analyze such games.

We will now arrive at a different notion, by asking a converse question: instead of asking which strategies a rational player will not play, we ask which strategies a rational player may consider playing. It turns out that this perspective can lead us to a notion of "solution" in many games where IESDS does not.

1 Motivating example

Consider the following two-player game:

	L	M	R
U	$(2,3)$	$(1,4)$	$(3,1)$
C	$(4,5)$	$(2,1)$	$(1,6)$
D	$(1,0)$	$(4,3)$	$(0,2)$

It is easy to verify that no strategy is strictly dominated for either player, and the IESDS process does nothing.

Instead, consider the scenario where the two players ask a *game theorist* (or a decision analyst) to recommend to each of them a strategy to play in the game. What strategy profile *should* the analyst recommend to the two players?

A minimal requirement on a "good" recommendation for a strategy profile is that, if each player believes the other player will follow the recommended strategy, then they themselves should have no incentive to deviate from the recommendation. In more detail, suppose player 1 believes player 2 will play the strategy recommended to her. In this scenario, if player 1 is free to play any strategy, then player 1 *should* want to play the strategy recommended to her. If even this condition does not hold, then there is no hope that the recommended strategy profile will be realized in the game.

So, let's see which strategy profiles meet this requirement.

Suppose the game theorist suggest the strategy profile (U, L). If player 1 assumes player 2 will play L, it does not make sense for player 1 to play U, since playing C gets player 1 higher payoff. So player 1 will *deviate* and play C.

Similarly, if player 2 assumes player 1 will play U, player 2 is strictly better off playing M. Thus, neither player will follow their suggestion, and both will deviate. In this case, we say the suggestion (U, L) is not reasonable, since it is not *stable* to deviations by players.

Next, consider (U, M). Here P2 will stick to the suggestion M if she assumes P1 will play U. But player 1, if she assumes P2 plays M, will be strictly better off playing D. Thus, again the suggestion (U, M) is not stable to deviations.

Performing the same analysis for all the strategy profiles, we have:

$$
\begin{aligned}
(U, L) \to\ & \text{P1 deviates to } C\text{, and P2 deviates to } M \to \text{not stable} \\
(U, M) \to\ & \text{P1 deviates to } D\text{, but P2 does not deviate} \to \text{not stable} \\
(U, R) \to\ & \text{P1 does not deviate, but P2 deviates to } M \to \text{not stable} \\
(C, L) \to\ & \text{P1 does not deviate, but P2 deviates to } R \to \text{not stable} \\
(C, M) \to\ & \text{P1 deviates to } D\text{, and P2 deviates to } R \to \text{not stable} \\
(C, R) \to\ & \text{P1 deviates to } U\text{, but P2 does not deviate} \to \text{not stable} \\
(D, L) \to\ & \text{P1 deviates to } C\text{, and P2 deviates to } M \to \text{not stable} \\
(D, M) \to\ & \text{Neither player deviates} \to \text{stable!} \\
(D, R) \to\ & \text{P1 deviates to } U\text{, and P2 deviates to } M \to \text{not stable.}
\end{aligned}
$$

Thus, we see that if any other strategy profile other than (D, M) is recommended, then at least one player will want to deviate, despite assuming that the other player will follow their recommendation. On the other hand, if the strategy profile (D, M) is recommended, then if P1 assumes P2 will play M, then P1 will play D. And if P2 assumes P1 will play D, then P2 will play M. Thus (D, M) is a strategy profile that meets our earlier requirement for a good recommendation.

We can consider this strategy profile (D, M) to be a solution of the game. Indeed, this concept is referred to as the *pure Nash Equilibrium*, after John Nash who introduced the concept.

The main ingredients in the notion of Nash Equilibrium are two fold:

1. Each player assumes the other player(s) will stick to their strategy recommendations.

2. Each player is free to *unilaterally deviate*, i.e., move to a different strategy (on their own). In other words, the players have not committed or contracted to playing any strategies beforehand. In particular, a rational player will deviate to a strategy if it gets her higher payoff.

With these two ingredients we were able to obtain a solution to the above two player game, in which the dominance notion did not yield a solution.

2 Formal definition and best reponse

Next, we will formalize the notion of Nash equilibrium. Given a game with N players with player i's pure strategy set S_i. Then we have the following:

Definition 1. *A pure strategy profile $(s_1, s_2, \ldots, s_N)$ is a pure Nash equilibrium (NE) of the game if for each i, we have*

$$\pi_i(s_i, s_{-i}) \ \geq \ \pi_i(s_i', s_{-i}) \qquad \textit{for all other } s_i' \in S_i.$$

i.e. if player i's strategy s_i maximizes player i's payoff $\pi_i(s_i', s_{-i})$ over all strategies s_i', assuming everyone else is playing according to s_{-i}.

We can see the two ingredients of Nash equilibrium in the above definition: (1) "Everyone else playing according to s_{-i}" is the first ingredient, and (2) "player i's strategy s_i maximizes her payoff π_i" is the second ingredient.

To capture the notion of player i maximizing her payoff, fixing s_{-i}, we develop the notion of a best response of player i.

Definition 2. *The* best response map *for player i is defined as follows: for any $s_{-i} \in \mathcal{S}_{-i}$*

$$\begin{aligned} \mathrm{BR}_i(s_{-i}) &= \text{ set of all strategies that maximize player } i\text{'s payoff when others play according to } s_{-i} \\ &= \left\{ s_i' : \pi_i(s_i', s_{-i}) \geq \pi_i(s_i'', s_{-i}) \text{ for all } s_i'' \right\}. \end{aligned}$$

For any fixed s_{-i}, we refer to $BR_i(s_{-i})$ as the best response set *of player i against s_{-i}.*

Note that $BR_i(s_{-i})$ contains *all* strategies for player i that maximize her payoff against s_{-i}. In particular, if there are multiple strategies that maximize the payoff, then $BR_i(s_{-i})$ contains all the maximizers.

In addition to the above best response maps for each player, we can define a best response correspondence that maps each strategy profile to a set of strategy profiles:

Definition 3. *The* best response correspondence *is defined as follows: for each $s = (s_1, s_2, \dots, s_N) \in \mathcal{S}$,*

$$\begin{aligned} BR(s) &= \{(s_1', s_2', \dots, s_N') : s_i' \in BR_i(s_{-i}) \text{ for each } i\} \\ &= BR_1(s_{-1}) \times BR_2(s_{-2}) \times \cdots \times BR_N(s_{-N}). \end{aligned}$$

In other words, the best response correspondence maps each strategy profile $s = (s_1, \dots, s_N)$ to a set of strategy profiles that could arise if each player i assumes everyone else will play s_{-i} and chooses a best response. If this is the case, then player 1 will choose some s_1' from the set $BR_1(s_{-1})$, player 2 will choose some s_2' from the set $BR_2(s_{-2})$ and so on.

Using the definition of best response maps and the best response correspondence, we can restate the definition of a pure NE equivalently as follows: a strategy profile $(s_1, s_2, \dots, s_N)$ is a pure Nash equilibrium if and only if

$$s_i \in \mathrm{BR}_i(s_{-i}) \qquad \text{for all } i,$$

i.e., the strategy s_i corresponding to each player i belongs to the best response set $\mathrm{BR}_i(s_{-i})$.

Equivalently, the strategy profile $s = (s_1, s_2, \dots, s_N)$ is a pure NE if and only if $s \in BR(s)$. Because the best response correspondence BR keeps any pure Nash equilibrium s fixed, i.e., $s \in BR(s)$, we say the pure Nash equilibrium s is a *fixed point* of BR.

Example. Consider again the game we analyzed earlier.

	L	M	R
U	$(2,3)$	$(1,4)$	$(3,1)$
C	$(4,5)$	$(2,1)$	$(1,6)$
D	$(1,0)$	$(4,3)$	$(0,2)$

For this game, we can write down the best response map for each player as follows.

$$\begin{aligned} \mathrm{BR}_1(L) &= \{C\}, & \mathrm{BR}_2(U) &= \{M\}, \\ \mathrm{BR}_1(M) &= \{D\}, & \mathrm{BR}_2(C) &= \{R\}, \\ \mathrm{BR}_1(R) &= \{U\}, & \mathrm{BR}_2(D) &= \{M\}. \end{aligned}$$

From these best response sets, we notice that $M \in \mathrm{BR}_2(D)$ and $D \in \mathrm{BR}_1(M)$. Thus (D, M) is a pure NE. For every other strategy profile $(s_1, s_2) \neq (D, M)$, we notice that either $s_1 \notin BR_1(s_2)$ or $s_2 \notin BR_2(s_1)$ (or both).

Using the best response maps, we write the best response correspondence as

$$\begin{aligned}
BR(U, L) &= \{(C, M)\} \\
BR(U, M) &= \{(D, M)\} \\
BR(U, R) &= \{(U, M)\} \\
BR(C, L) &= \{(C, R)\} \\
BR(C, M) &= \{(D, R)\} \\
BR(C, R) &= \{(U, R)\} \\
BR(D, L) &= \{(C, M)\} \\
BR(D, M) &= \{(D, M)\} \\
BR(D, R) &= \{(U, M)\}.
\end{aligned}$$

From this we see that $(D, M) \in BR(D, M)$, and hence, again, (D, M) is a pure Nash equilibrium. No other strategy s satisfies $s \in BR(s)$.

In general, if $s_i \notin \mathrm{BR}_i(s_{-i})$, then there is some other strategy that gets player i higher payoff and hence (s_i, s_{-i}) cannot be a pure NE. We can use this to create an *algorithm* to find all the pure NE in a game.

1. For each player i, compute $\mathrm{BR}_i(s_{-i})$ for each $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$.
2. For each pure strategy profile $s = (s_1, s_2, \ldots, s_N)$ check if $s_i \in \mathrm{BR}_i(s_{-i})$ for all i; if yes, then $s = (s_1, \ldots, s_N)$ is a pure NE. On the other hand, if there is some i such that $s_i \notin BR_i(s_{-i})$ then s is not a pure NE.

Example. Consider the following two player game:

	L	M	Q
U	$(2,3)$	$(3,0)$	$(1,3)$
D	$(3,0)$	$(4,3)$	$(2,3)$

First, we find the best response maps.

$$\begin{aligned}
&\mathrm{BR}_1(L) = \{D\}, && \mathrm{BR}_1(M) = \{D\}, && \mathrm{BR}_1(Q) = \{D\} \\
&\mathrm{BR}_2(U) = \{L, Q\}, && \mathrm{BR}_2(D) = \{M, Q\}.
\end{aligned}$$

Next, note that $U \notin \mathrm{BR}_1(s_2)$ for any $s_2 \implies U$ cannot be part of pure NE. For other strategy profile, we check

$$\begin{aligned}
(D, L): &\quad D \in \mathrm{BR}_1(L) \text{ but } L \notin \mathrm{BR}_2(D) \implies (D, L) \text{ is not a pure NE} \\
(D, M): &\quad D \in \mathrm{BR}_1(M) \text{ and } M \in \mathrm{BR}_2(D) \implies (D, M) \text{ is pure NE.} \\
(D, Q): &\quad D \in \mathrm{BR}_1(Q) \text{ and } Q \in \mathrm{BR}_2(D) \implies (D, Q) \text{ is pure NE.}
\end{aligned}$$

Thus, in the above game, we see that there are two pure Nash equilibria.

Example (Matching pennies)**.** Consider the matching pennies game with payoffs as follows:

	H	T
H	$(1,-1)$	$(-1,1)$
T	$(-1,1)$	$(1,-1)$

We have

$$\begin{aligned} \mathrm{BR}_1(H) &= \{H\}, \qquad \mathrm{BR}_1(T) = \{T\} \\ \mathrm{BR}_2(H) &= \{T\}, \qquad \mathrm{BR}_2(T) = \{H\}. \end{aligned}$$

We again perform the check for each strategy profile:

$$\begin{aligned} (H,H):& \quad H \notin \mathrm{BR}_2(H) \implies (H,H) \text{ is not a pure NE} \\ (H,T):& \quad H \notin \mathrm{BR}_1(T) \implies (H,H) \text{ is not a pure NE} \\ (T,H):& \quad T \notin \mathrm{BR}_1(H) \implies (H,H) \text{ is not a pure NE} \\ (T,T):& \quad T \notin \mathrm{BR}_2(T) \implies (T,T) \text{ is not a pure NE.} \end{aligned}$$

Thus, we conclude that there is no pure Nash equilibrium in the game of matching pennies.

From the above two examples, we see that a game may have more than one pure NE or no pure NE at all.

Although this “algorithm” can find all pure NE that exist, it is too cumbersome as it checks over each pure strategy profile. As we have seen before, even a small game can have large number of strategy profiles. The computation of Nash equilibrium is an issue we will address later.

Next, we study how pure NE are related to the previous solution concepts we have considered, namely, strict DSE, weak DSE, solution at the end of IESDS, etc.

Relating pure Nash equilibrium to dominance notions

1. Can a strictly dominated strategy be part of a pure Nash Equilibrium?

   No. If a strategy is strictly dominated, that means there is another strategy that gets higher payoff no matter what others do. Thus, such a strategy can never belong to a best response set $\mathrm{BR}_i(s_{-i})$ for any s_{-i}.

2. If a game has a strict DSE, then such a strategy profile must be a Nash equilibrium.

   A strictly dominant strategy is the unique best response to any strategy profile of the opponents. Thus a strictly dominant strategy must be part of a NE.

3. If a game has a strict DSE, there cannot be another pure NE. (Exercise: why?)

4. If a game is dominance solvable, i.e. if IESDS leads to a single strategy profile, that profile must be a pure Nash equilibrium. Why? *Hint:* Every other strategy is strictly dominated in some reduced form game.

5. The IESDS process will never delete a strategy that is part of a pure Nash Equilibrium.

This above relationship is summarized in Figure 2

What about the relation between NE and weak dominance? This is more nuanced that strict dominance, as the following game illustrates.

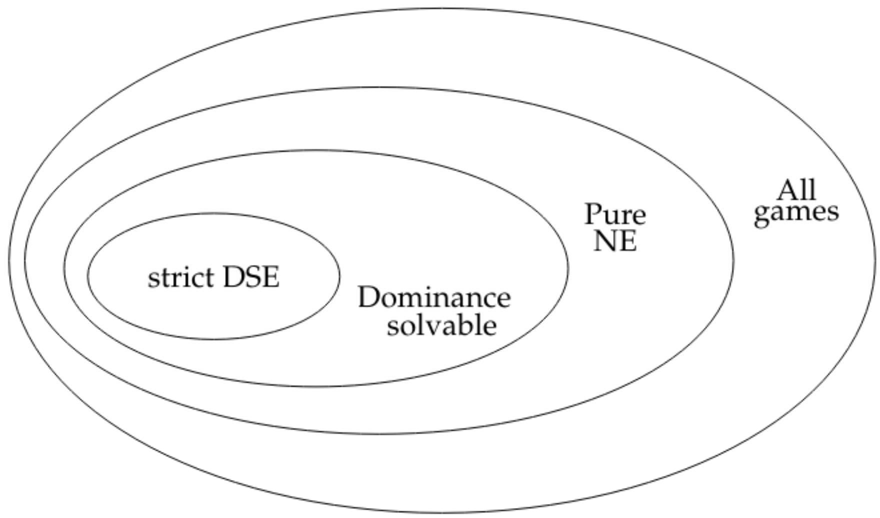

Figure 1: Games with strict DSE $\subset$ Dominance solvable games $\subset$ Games with pure NE $\subset$ All games

	L	M	Q
U	$(2,3)$	$(3,0)$	$(1,3)$
D	$(3,0)$	$(4,3)$	$(2,3)$

We have the following observations:

1. Weakly dominated strategy can be part of pure NE!

   M is weakly dominated by Q. But (D, M) is a pure NE.

2. If a game has weak DSE, it is a pure NE.

   (D, Q) is a weak DSE and hence also a pure NE.

3. If a game has a weak DSE, it need not be the unique NE.

   (D, Q) is a weak DSE (and a pure NE), but so is (D, M).

4. Removing weakly dominated strategies may delete pure NE in a game.

   M is weakly dominated by Q, so if we eliminate M from the game, we would also eliminate the NE (D, M).

The above nuanced relationship is another reason why we need to be cautious in eliminating weakly dominated strategies.