Static Games of Complete Information

IE 5545: Decision Analysis
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Having developed the (Bayesian) decision making framework, we now extend it to games, i.e., settings with more than one decision makers (commonly referred to as "players"). We begin by focusing on games that involve all players making a single decision simultaneously and independently, and the actions of all the players together determining the outcome and the payoffs. We call such games as "static games". Furthermore, we will focus on the setting where there is no incomplete information — meaning all players know the complete details of the game, and each player has the same information.

For example, in chess, both players observe the board and all the pieces at all times. Hence chess is a game of complete information. However, since the moves are performed sequentially, it is dynamic game of complete information. Similarly, in a sealed-bid auction, each player ("bidder") must submit a secret bid for the auctioned item without knowledge of the others' bids. Thus, it is static game. However, while a bidder typically knows their value for the auctioned item, they do not know how much others value it. Thus an auction is a static game of incomplete information.

Some simple examples of static games of complete information include matching pennies, rock-paper-scissors, and prisoner's dilemma.

Summarizing, static games of complete information have the following features (1) each player chooses a single action independently and simultaneously and (2) the actions of all the players together determines the outcome and each player's payoff.

1 Strategic form

Consider a game with with N players, represented by the set $\{1, 2, \ldots, N\}$. Let $\mathcal{S}_i$ denote the set of strategies available to player i, with $s_i \in \mathcal{S}_i$ denoting a particular strategy. A strategy profile for the game is a "vector" comprised of a single strategy for each player:

$$s = (s_1, s_2, s_3, \ldots, s_N).$$

where s_i is a strategy for player i. We let $\mathcal{S}$ denote the set of all strategy profiles:

$$\mathcal{S} = \{(s_1, s_2, \ldots, s_N) : s_1 \in \mathcal{S}_1,\ s_2 \in \mathcal{S}_2,\ \ldots,\ s_N \in \mathcal{S}_N\}.$$

Note that $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_N$, the Cartesian product of the sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N$.

Each strategy profile $s = (s_1, s_2, \ldots, s_N) \in \mathcal{S}$ induces a distribution over outcomes. This outcome distribution determines the expected utility of each player when the strategy profile s is played in the game. This leads us to the notion of a payoff function. The payoff function π_i for player i takes as input a strategy profile s and gives as output player i's expected utility when each player plays according to s.

It is often desirable when writing the payoff function π_i to separate the dependence on player i's strategy and the strategies of everyone else. For this purpose, we will often write the payoff of player i under strategy profile $s = (s_1, s_2, \ldots, s_n) \in \mathcal{S}$ as

$$\pi_i(s) = \pi_i(s_i, s_{-i})$$

where $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$ denotes the strategies of all players other than player i.

Summarizing, this preceding description of a game is called its *strategic form*, or sometimes its *normal form*, and involves listing the following information

1. the set of players: $\mathcal{N} = \{1, 2, \ldots, N\}$.
2. the set of strategies for each player: $\mathcal{S}_i = \{1, 2, \ldots, k_i\}$ for each $i \in N$.
3. the payoff functions of all players: $\pi_i : \mathcal{S} \to \mathbb{R}$ for each i, where $\mathcal{S}$ is the set of all strategy profiles.

2 Rationality and common knowledge

Just as in the decision theory framework, we make the assumption that each player is rational, and knows the structure of the game, namely its strategic form. Call this assumption as (0).

(0) Each player is rational, and knows the structure of the game.

However, while playing a game, a rational player has to reason about how other players might play the game. For this, a player i must assume that another player $j \neq i$ is rational and knows the structure of the game. This leads us to the assumption (1):

(1) Each player knows that all players are rational and know the structure of the game.

But then, in order for a player $k \neq i, j$ to reason about the game, she must reason about how player i reasons about player j. This gets us assumption (2):

(2) Each player knows each player knows that all players are rational and know the structure of the game.

We can keep going down higher levels of what player i knows about player j knows about player k... and so on. Each level $n \geq 0$ will give us an assumption (n).

(n) Each player knows ... (n times) ... that all players are rational and know the structure of the game.

We refer to this *infinite* set of assumptions as "common knowledge". To say some information I is common knowledge means that all statements of the form "i knows that j knows that ... k knows I", of any length, is true.

Thus, we make the assumption that

(CK) Rationality of all players and the structure of the game is common knowledge.

3 Two player games

When $N = 2$, i.e. two players, again we can list out π_1 & π_2 as matrices, instead of functions. Suppose

$$\mathcal{S}_1 = \{1, 2, \ldots, m\}, \qquad \mathcal{S}_2 = \{1, 2, \ldots, n\}.$$

The payoff functions can be represented as two matrices Π_1 and Π_2, where player 1's strategies are listed as rows and player 2's strategies are listed as columns, and the cell entries to row i and column j correspond to the players payoff under strategy profile (i, j):

$$\Pi_1 = \begin{array}{c|cccc} & 1 & 2 & \cdots & n \\ \hline 1 & \pi_1(1,1) & \pi_1(1,2) & \cdots & \pi_1(1,n) \\ 2 & \pi_1(2,1) & \pi_1(2,2) & \cdots & \pi_1(2,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m & \pi_1(m,1) & \pi_1(m,2) & \cdots & \pi_1(m,n) \end{array} \qquad \Pi_2 = \begin{array}{c|cccc} & 1 & 2 & \cdots & n \\ \hline 1 & \pi_2(1,1) & \pi_2(1,2) & \cdots & \pi_2(1,n) \\ 2 & \pi_2(2,1) & \pi_2(2,2) & \cdots & \pi_2(2,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m & \pi_2(m,1) & \pi_2(m,2) & \cdots & \pi_2(m,n) \end{array}$$

Due to this, a two-player game is sometimes referred to as a *bimatrix game*. The two matrices are often represented more compactly as a single table, where the first entry denotes P1's payoff and second denotes P2's payoff when the corresponding strategy profile is played.

Player 1	1	2	$\cdots$	n
1	$(\pi_1(1,1), \pi_2(1,1))$	$(\cdot,\cdot)$	$\cdots$	$(\cdot,\cdot)$
2	$(\pi_1(2,1), \pi_2(2,1))$	$(\cdot,\cdot)$	$\cdots$	$(\cdot,\cdot)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
m	$(\cdot,\cdot)$	$(\cdot,\cdot)$	$\cdots$	$(\cdot,\cdot)$

Player 2

Example (Matching pennies)**.** In the matching pennies game, two players choose between heads (H) and tails (T), where player 1 wants to match with player 2, whereas player 2 wants to mismatch. Assuming a win yields a utility of 1 for each player and a loss yields a utility of -1, the strategic form of matching pennies can be represented with payoffs as

	H	T
H	$(1,-1)$	$(-1,1)$
T	$(-1,1)$	$(1,-1)$

i.e.

$$\pi_1(H,H) = \pi_1(T,T) = 1 \qquad \pi_1(H,T) = \pi_1(T,H) = -1$$
$$\pi_2(H,H) = \pi_2(T,T) = -1 \qquad \pi_2(H,T) = \pi_2(T,H) = 1.$$

Example (Rock-paper-scissors)**.** Recall that rock (R) beats scissors (S), scissors beats paper (P) and paper beats rock. Assuming a win yields a utility of 1, a loss yields a utility of -1 and a draw yields a utility of 0, we have the following strategic form representation:

	R	P	S
R	$(0,0)$	$(-1,1)$	$(1,-1)$
P	$(1,-1)$	$(0,0)$	$(-1,1)$
S	$(-1,1)$	$(1,-1)$	$(0,0)$

Our goal over the next few lectures will be to understand how to analyze a static game of complete information, and obtain a prediction about how players will play in such a game. We will make our way slowly towards a general solution concept, by building our intuition through the analysis of simple games. We begin our journey with the classic game: Prisoner's Dilemma.

4 Prisoner's dilemma

This is a classic two player game with the following back-story:

Two individuals A and B have been arrested on some minor offense. The police suspect they are part of a larger criminal operation, but do not have any evidence of this that will stand in a court of law. Their only recourse is if at least one of the two individuals confess to the larger crime. To accomplish this, the police separate the two individuals in two different cells, allowing no communication between the two, and ask each to admit to the larger crime with the following offer.

1. If both individuals admit to the crime, then they each get five years of jail term.
2. If A admits to the crime, and B does not then A will be considered as an informant against B, and A will be set free, and B will face the full impact of the law and serve 10 years in jail.
3. Vice versa if B admits and A stays silent.
4. If both stay silent, then they will only be charged with the lesser offense and will serve only 1 year in jail each.

The question goes: how should individuals A and B act, if both are rational? In particular, if you are A, would you stay silent, or would you betray your partner?

To analyze this game, we first write down the strategic form. Assuming each individual's utility is equal to negative of number of years they spend in jail, we obtain the following:

	stay silent (C)	betray (D)
stay silent (i.e., cooperate) (C)	$(-1,-1)$	$(-10,0)$
betray (i.e., defect) (D)	$(0,-10)$	$(-5,-5)$

Remark 1. *Note "cooperate" and "defect" are common terminology used while studying prisoner's dilemma. In this context, a player cooperates with her partner by staying silent, whereas she defects by betraying her partner.*

From the strategic form, we observe that if player B plays C, then player A gets strictly higher payoff when playing D, i.e. betraying. $(0 \text{ vs } -1)$.

Similarly, if player B plays D, i.e. betrays, again player A gets strictly higher payoff when playing D $(-5 \text{ vs } -10)$.

Thus, irrespective of how player B plays, player A gets strictly higher payoff on playing D than on playing C.

In this case, it is reasonable to expect that a rational player A will never play C. By symmetry, a rational player B will also never play C.

Thus, the only outcome that is plausible is (D, D).

Thus, in the prisoner's dilemma, both individuals will betray their partners and confess to the larger crime.

Note that both individuals would have been better off if they could have coordinated and stayed silent. But even if they did, one of them would have preferred to deviate and betray.

From the analysis of prisoner's dilemma, we can infer the underlying intuition that a rational player will never play a strategy if there is some other strategy that always gets strictly higher payoff. We formalize this intuition by the notion of strict dominance.

Definition 1 (Strictly dominated strategy). *A strategy s_i for player i is* strictly dominated *by another strategy s_i' if*

$$\pi_i(s_i, s_{-i}) < \pi_i(s_i', s_{-i}) \qquad \text{for all } s_{-i},$$

i.e. playing s_i' gets strictly higher payoff than s_i, no matter what strategies s_{-i} other players play. In this case, we say strategy s_i' strictly dominates *strategy s_i.*

For example, in prisoner's dilemma, stay silent (C) is strictly dominated by betray (D).

Naturally, a rational player will never play a strictly dominated strategy s_i, since the player can always play the strategy s_i' that dominates s_i instead and get strictly higher payoff.

Definition 2 (Strictly dominant strategy). *A strategy s_i is strictly dominant if any other strategy s_i' is strictly dominated by s_i.*

For example, Betray (D) is strictly dominant in Prisoner's dilemma.

Based on our earlier observation that a rational player will never play a strictly dominated strategy, we conclude that if a rational player has a strictly dominant strategy s_i, then the player will play strategy s_i in the game.

Finally, if all players have a strictly dominant strategy, then the only possible outcome is each player playing their strictly dominant strategy. This motivates the notion of a strict dominant strategy equilibrium (DSE):

Definition 3 (Strict dominant strategy equilibrium (DSE)). *A strategy profile $S = (s_1, s_2, \ldots, s_N)$ is a strict DSE of a N-player game, if and only if s_i is strictly dominant for player i, for each i.*

Thus, in prisoner's dilemma, (D, D) is a strict DSE.

A strict DSE is a strong predictor of how players will play in a game. The main advantage of a strict DSE is that players do not have to reason about how others play. All one has to assume is that rational players will not play strictly dominated strategies, which is a weak assumption. We never used the higher levels of the common knowledge assumption to arrive at the notion of strict DSE.

However, most games do not have a strict DSE. In fact, a game may not have any strictly dominated strategies at all. That said, if one is designing a game, if it is possible to design it so that it has a strict DSE, then the actions of the participants can be perfectly predicted (as long as they are rational).

5 Iterated strict dominance

Now, consider the following game:

	L	M	R
U	$(2,3)$	$(3,0)$	$(2,1)$
C	$(3,0)$	$(4,3)$	$(5,5)$
D	$(5,1)$	$(2,2)$	$(4,1)$

In this game, U is strictly dominated by C, so player 1 will not play U. However, there are no other strictly dominated strategies. So if we stop at the intuition that a rational player will not play a strictly dominated strategy, we are stuck and do not have a solution to this game.

However, we can make progress if we assume that player 2 knows that player 1 is rational and will not play strategy U. In that case, since both players know that strategy U will never be played, we can strike it out to get a smaller game ("reduced game"):

	L	M	R
C	$(3,0)$	$(4,3)$	$(5,5)$
D	$(5,1)$	$(2,2)$	$(4,1)$

In this reduced game, P1 has only two strategies C and D. Now, we see that in this game L is strictly dominated by M. Thus a rational player 2 will never play L in this reduced game.

We can thus say that a rational player 2 who knows that player 1 is rational will never play L in the original game. The reason is that P2 knows P1 is rational and hence will not play U, and given P1 doesn't play U, strategy L is strictly dominated by M.

We can continue further if P1 knows P2 is rational and knows P1 is rational. In that case, P1 knows P2 will not play L. So the further reduced game has the following strategic form:

	M	R
C	$(4,3)$	$(5,5)$
D	$(2,2)$	$(4,1)$

In this reduced game, D is strictly dominated by C. Again P1 will not play D in this reduced game, and hence a rational P1, who knows that P2 knows that P1 is rational, will not play U and D. And a rational P2 who knows that P1 knows that P2 knows that P1 is rational will consider the following reduced game

	M	R
C	$(4,3)$	$(5,5)$

and choose R, since M is strictly dominated by R.

Thus, the final reduced game is

	R
C	$(5,5)$

and hence the solution is (C, R) with payoff $(5, 5)$.

In general, we can perform this iterated analysis because we assume common knowledge of rationality. Thus, if we assume that both players know the other is rational, know that others know that they are rational etc, then we can iteratively eliminate successive strictly dominated strategies, and analyze reduced games.

This procedure we considered is called "iterated elimination of strictly dominated strategies" (IESDS) or "iterated strict dominance" (ISD).

If a game G has the property that at the end of the ISD process we are left with a single strategy profile, then we say G is "dominance solvable" and call the resulting strategy profile a *solution* to the game G or sometimes as an *ISD-equilibrium*.

1. IESDS requires as many steps of "P1 knows P2 knows ... is rational", as the number of steps required for the ISD process to end, i.e., until there are no more strictly dominated strategies.

2. IESDS is less powerful than a strict DSE as a solution concept, because it requires much more than rationality of players, i.e., it requires in general common knowledge of rationality (see the preceding point).

3. Just like the case with strict DSE, many games are not dominance solvable: at the end of IESDS process, we are still left with a game with multiple strategy profiles. IESDS is ineffective as a tool for prediction of game play in this instance.

6 Weak dominance

Consider the following game.

	L	M	Q
U	$(2,3)$	$(3,0)$	$(1,3)$
D	$(3,0)$	$(4,3)$	$(2,3)$

In this game, L and M are not strictly dominated by Q. So based on our intuition that a rational player will not play strictly dominated strategy, we cannot eliminate any strategy for player 2.

However, we see that playing Q always gets at least as much payoff as L or M. Further, there is at least one strategy of player 1 against which play Q is strictly better than play L. (Similarly for M). In such a case, we say L (and M) are weakly dominated by Q.

Definition 4 (Weakly dominated strategy)**.** *A strategy s_i for player i is weakly dominated by s_i' if*

$$\pi_i(s_i, s_{-i}) \leq \pi_i(s_i', s_{-i}) \qquad \text{for all } s_{-i}$$

and there is at least one s_{-i} such that

$$\pi_i(s_i, s_{-i}) < \pi_i(s_i', s_{-i}).$$

We can similarly define a weakly dominant strategy as a strategy s_i such that any other strategy s_i' is weakly dominated by s_i. Also, a weak DSE is a strategy profile $s = (s_1, \ldots, s_N)$ such that each s_i is a weakly dominant strategy for player i.

Note that a strictly dominant strategy is also weakly dominant, and hence a strict DSE is also as weak DSE. The converse is not true in general.

Assuming rational players will not play a weakly dominated strategy is not as robust as the assumption that they will not play a strictly dominated strategy. This is because, for e.g., the one strategy profile s_{-i} for other players at which a weakly dominated strategy s_i is strictly worse than s_i' may be strictly dominated for other players. Consider the game we analyzed

	L	M	Q
U	$(2,3)$	$(3,0)$	$(1,3)$
D	$(3,0)$	$(4,3)$	$(2,3)$

Here M is weakly dominated by Q, because

$$\pi_2(U, M) < \pi_2(U, Q) \qquad\qquad \pi_2(D, M) = \pi_2(D, Q).$$

But the strategy of player 1 at which Q is better than M, i.e., strategy U, is itself strictly dominated by D. So if rationality is common knowledge, then both players know player 1 will not play U and the reduced game is

	L	M	Q
D	$(3,0)$	$(4,3)$	$(2,3)$

and now, in this game, M is no longer weakly dominated by Q, and we can imagine a rational player 2 playing M.

The above example also shows that if we performed iterated elimination of weakly dominated strategies, then the reduced game we will end up in could depend on the order in which one eliminates the strategies. For instance, if we first eliminate player 2's weakly dominated strategies and then do player 1's, then we end up with (D, Q). On the other hand, if we first eliminate player 1's weakly dominated strategies, and then do player 2's, then we end up with $\{(D, Q), (D, M)\}$. This is in contrast with IESDS, where the order of elimination does not matter.

Although eliminating weakly dominated strategies is not justified in some cases and may eliminate reasonable strategy profiles, it is still used in practice to come up with solutions to games, especially if the resulting strategy profile has a nice structure. For example, in a second price auctions, truthful bidding, i.e., submitting a bid that equals ones value for the item, is a weak-DSE.