Worksheets

IE 5545: Decision Analysis
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Inventory decision: Consider a restaurant owner who must decide how many walleyes to purchase every day for the restaurant's signature dish, a whole walleye fish fry. Each walleye costs $c = \$8$ dollars. Each order of the signature dish sells for $p = \$12$. Any leftover fish at the end of the day is used up in making the fish soup the next day, which results in a revenue of $s = \$4$ per fish.

The demand for the signature dish is distributed as a $\mathsf{Geom}_{\geq 0}(p)$ random variable with success probability $p = 0.01$.

How many walleyes should the restaurant purchase each day, if its goal is to maximize the expected profits? (Assume labor & staffing costs, overheads, etc are already incurred and do not depend on the decision.)

Note: A $\mathsf{Geom}_{\geq 0}(p)$ random variable X with success probability p has the probability mass function

$$\mathbf{P}(X = k) = (1-p)^k p, \quad \text{for } k = 0, 1, 2, \ldots.$$

Pricing decision, a.k.a "A monopolist who cares": A monopolist must decide at what price to sell its product to a group of consumers. Assume the monopolist's marginal cost of production is negligible.

Each consumer has a valuation v for the product that is drawn independently and identically from an $\mathsf{Expo}(\lambda)$ distribution with rate $\lambda = 2$.

The monopolist cannot observe the consumers' valuation and hence must set a single price p for the product. If the price for the product is $p \geq 0$, then only those consumers with valuation $v > p$ purchase the product.

The monopolist, with a long term view to retain its market share, cares not just about its profits, but also about consumer surplus, defined as follows: if a consumer with valuation v purchases the product, the surplus is $v - p$; otherwise the surplus is 0.

Specifically, the monopolist puts twice as much weight on its profits as on the consumer surplus. How should the monopolist set the price?

Note: A random variable $X \sim \mathsf{Expo}(\lambda)$ has the probability density given by

$$f(x) = \lambda e^{-\lambda x} \quad \text{for all } x \geq 0.$$

Pricing with uncertainty. Consider a seller who must decide the price for an item. The seller incurs a unit cost $c = 0.25$ for the item, and wants to set a price to maximize the profit, given by $(p - c) \cdot D(p)$, where $D(p)$ is the demand at price p. However, the seller is uncertain about the demand for the item. The decision maker knows that either the item will be a hit, or it will be a flop. If the item is a hit, the demand for the item at price p is given by $D_{\mathsf{hit}}(p) = \max\{100 - 25p, 0\}$. If the item is a flop, the demand for the item is $D_{\mathsf{flop}}(p) = \max\{80 - 40\sqrt{p}, 0\}$.

In this example, the set of states is finite and is given by $\Theta = \{\mathsf{hit}, \mathsf{flop}\}$. Note that in either state, the demand is zero if $p > 4$. The seller believes that that the item is twice as likely to be a hit as it is likely to be a flop:

$$\mu(\mathsf{hit}) = \mathbf{P}(\bar{\theta} = \mathsf{hit}) = \frac{2}{3} \qquad\qquad \mu(\mathsf{flop}) = \mathbf{P}(\bar{\theta} = \mathsf{flop}) = \frac{1}{3}.$$

(a) Formulate the seller's problem in the Bayesian decision theory framework: In addition to Θ and μ, state the set of actions $\mathcal{S}$, set of outcomes $\mathcal{O}$, the (state-dependent) rules $\bar{p}_\theta$, the (state-dependent) utility function u_θ and the payoff function $\pi(\theta, s)$.

(b) State the seller's expected payoff function $\Pi(s)$.

(c) Find the optimal price for the seller. [Skip]

Retail promotion. Consider an online retailer who must decide whether to offer a user a discount on an item being sold on their website. The regular price for the item is given by $r = \$10$. If the discount is offered, the price is $d = \$8$. The discount has the effect of increasing the customer's probability of purchasing the item: if the customer purchases the item at a regular price with probability $p \in [0, 1]$, then with the discount, she will purchase it with probability $H(p) = \frac{1+p}{2} \geq p$. The retailer wants to maximize her expected revenue R, where $R = 10p$ if no discount is offered, and $R = 8H(p) = 4(1 + p)$ if a discount is offered. However, the retailer does not know the probability p with which the user will purchase an item without a discount.

In this example, the set of states is all the possible values of $p \in [0, 1]$. If the retailer has no information about p, then the set of states is infinite and is given by $\Theta = [0, 1]$. Suppose the retailer's belief μ is specified by the distribution with density $f(x) = 6x(1 - x)$ for $x \in [0, 1]$. (Under this belief, values close to 0.5 are more likely compared to either extremes.)

(a) Formulate the retailer's problem in the Bayesian decision theory framework: In addition to Θ and μ, state the set of actions $\mathcal{S}$, set of outcomes $\mathcal{O}$, the (state-dependent) rules $\bar{p}_\theta$, the (state-dependent) utility function u_θ and the payoff function $\pi(\theta, s)$.

(b) State the retailer's expected payoff function $\Pi(s)$.

(c) Find the optimal decision for the retailer.

Cournot competition. We consider the model of oligopoly competition studied by Cournot (1838). The model analyzes a firm's decision problem about determining the production quantity of a commodity, when it faces competition from other firms in the market.

Consider an oligopoly with two firms $i \in \{1,2\}$, who each produce a single homogeneous commodity. Each firm i must decide her production level $q_i \geq 0$. The cost of production for firm i is given by $c_i(q_i)$. Once the production levels are determined (and produced), the total quantity of the commodity $Q = q_1 + q_2$ will be sold in the market at the "market clearing" price $P(Q)$, where the market is said to clear when the total supply of goods is equal to the total demand at that price. (Hence $P(\cdot)$ is also called the *inverse demand function*.) Each firm seeks to maximize her profit, which is her revenue minus her cost of production:

$$\begin{aligned}\pi_i(q_i, q_{-i}) &= q_i P(Q) - c_i(q_i) \\ &= q_i P(q_i + q_{-i}) - c_i(q_i).\end{aligned}$$

We assume the functions $P(\cdot)$ and $\{c_i(\cdot) : i \in \{1,2\}\}$ are commonly known to both the firms. Thus, we obtain a strategic form game with complete information.

To make things little simpler, we make some further assumptions on the model:

1. The inverse demand function $P(Q)$ is given by

   $$P(Q) = (a - bQ)^+ = \max\{a - bQ, 0\}.$$

   In other words, the price is given by $a - bQ$ if $Q \leq \frac{a}{b}$, and 0 otherwise. (This assumption models the scenario where if the total quantity Q goes beyond a level $\frac{a}{b}$, the price in the market drops to a very low level.)

2. The cost of production for the two firms is given by

   $$c_i(q_i) = cq_i.$$

   This means that the cost of production is linear with no fixed costs, and the marginal cost of production is same for the two firms, and is given by c. (The assumption that the marginal cost of production is same for the two firms models the scenario where the two firms are using the same "technology" to produce the commodity.)

(a) Find the best response maps for each firm. The map $BR_i(q_{-i})$ for firm i should depend only on the quantity q_{-i} of the other firm.

(b) Find a pure Nash equilibrium of this game.

(c) Compare the outcome with what would occur if the two firms collude to maximize total profit.

Bayesian inference. Let $N(t)$ be the number of users arriving to a web publisher's website in time interval $[0, t)$. It is known that $N(t)$ is a Poisson process with an **unknown** arrival rate $\bar{\theta}$. That is, given $\bar{\theta} = x > 0$, the number of users $N(t)$ arriving to the website in the time interval $[0, t)$ is distributed as

$$\mathbf{P}(N(t) = k | \bar{\theta} = x) = e^{-xt} \frac{(xt)^k}{k!}, \quad \text{for } k = 0, 1, 2, \ldots.$$

You have been tasked with estimating the arrival rate $\bar{\theta}$. Your prior belief about $\bar{\theta}$ is given by a $\mathsf{Gamma}(1, 1)$ distribution. You plan to collect data about the number of arrivals $N(1)$ in the time interval $[0, 1)$, and use that to estimate $\bar{\theta}$.

Fact 1: A $\mathsf{Gamma}(\alpha, \beta)$ distribution, with parameters α and β, has a density $f_{(\alpha,\beta)}$ given by

$$f_{(\alpha,\beta)}(\bar{\theta} = x) = \frac{x^{\alpha-1} e^{-x\beta}}{G(\alpha, \beta)}, \quad \text{for all } x \geq 0.$$

Here, $G(\alpha, \beta)$ is a normalizing constant given by $G(\alpha, \beta) \triangleq \int_0^\infty x^{\alpha-1} e^{-x\beta} dx$. If α is a positive integer, then we have the following expression:

$$G(\alpha, \beta) = \frac{(\alpha - 1)!}{\beta^\alpha}$$

Fact 2: The mean of a $\mathsf{Gamma}(\alpha, \beta)$ distribution is equal to α/β.

(a) Suppose you observe $N(1) = 3$. What is your posterior belief about $\bar{\theta}$?

(b) Compute $\mathbf{E}[\bar{\theta} | N(1) = k]$ as a function of k.

(c) Compute $P(N(1) = k)$ as a function of k. What is this distribution?

Signal structures. There is a roulette wheel that takes 5 equally likely values A, B, C, D, E. Before the wheel spins, you decide whether to place a bet. If you place a bet, your net payoffs are

$$\text{net payoff} = \begin{cases} +6 & \text{if wheel lands on } A; \\ +3 & \text{if wheel lands on } B; \\ +2 & \text{if wheel lands on } C; \\ -4 & \text{if wheel lands on } D; \\ -6 & \text{if wheel lands on } E. \end{cases}$$

If you do not place bets, your payoffs are all 0. Suppose you have access to the following information before you decide. In each case, represent the information as a signal structure, and compute its value of information.

(a) Information that reveals whether the wheel will land on a vowel.

(b) Information that shares the word from the set $\{\mathsf{ALE}, \mathsf{BUD}, \mathsf{COORS}\}$ which contains the letter on which the wheel lands.

(c) Information that share the word from the set $\{\mathsf{CYBERPUNK}, \mathsf{HARDSF}\}$ which contains the letter on which the wheel lands.

(d) Information sharing both (b) and (c).

Bayesian decision framework. A plant manager must decide what to do with a critical pump before a high-demand week.

The pump can be in one of three condition: healthy, degrading or near-failure. The pump's condition affects the probability that the pump will fail during the high-demand week. If the pump is healthy, it will fail with probability 1% (and not fail with remaining probability). If the pump is degrading, it will fail with probabilty 10%. If it is near-failure, it will fail with probability 80%.

If the pump fails during the high-demand week, then the manager incurs a operational cost of $\$100,000$. If the pump does not fail, then the manager incurs no operational cost.

The manager believes that the pump is healthy with 60% probability, and it is degrading with 30% probability.

The manager can take one of three actions now: (1) do nothing, or (2) schedule a maintenance, or (3) preemptively replace the pump.

Doing nothing incurs no additional costs (beyond any costs due to pump failure).

Scheduling a maintenance requires stopping the pump for few hours and the downtime incurs a cost of $4000. However, the maintanence improves the pump's condition: if the pump was near-failure, its condition improves to degrading, and if the pump was degrading, its condition improves to healthy. The maintanence does not hurt a healthy pump.

Preemptively replacing the pump costs $18000. However, the new pump will be in healthy condition.

(a) Formulate the manager's decision problem in the Bayesian decision theory framework. Specifically, describe the states of the world, the prior belief, the set of actions, the set of outcomes, the rules, and the payoff function.

(b) What is the optimal action of the manager?

(c) Before making the decision, the manager decides to do a vibration test of the pump. The test outcome is either "pass" or "fail". A healthy pump passes the test with probability 95%. A degrading pump passes the test with probability 40%. A near-failure pump passes the test with 5%. What should be the plant manager's optimal actions after performing the test? (The optimal action will depend on the test outcome.)

(d) What is the value of performing the test?

Best response sets and pure NE. Consider a three-player game, where each player $i \in \{1, 2, 3\}$ chooses a number $x_i \in [0, \infty)$, and has the following payoff function:

$$\pi_i(x_1, x_2, x_3) = 2x_i \left(\sum_{j \neq i} x_j - i \right) - x_i^2.$$

(a) For each player i, determine the values of x_{-i} for which $x_i = 0$ is a best response (i.e., $0 \in BR_i(x_{-i})$).

(b) Compute the best response set $BR_i(x_{-i})$ for all values of x_{-i} and each player i.

(c) Find all pure Nash equilibria of this game.

Best response mappings Consider a two player game with the following payoffs.

	A	B
a	$(2,3)$	$(3,4)$
b	$(5,4)$	$(2,1)$

(a) State the best response maps for each player.

(b) Find all NE of this game.

Fully mixed Nash equilibrium. Consider a two player game with three strategies each. The payoff matrices for the two players are given by

	L	M	R
U	$(0,2)$	$(5,2)$	$(1,3)$
C	$(1,4)$	$(3,6)$	$(1,5)$
D	$(2,9)$	$(3,4)$	$(0,4)$

(a) What are the pure NE of this game?

(b) Find a fully mixed Nash equilibrium of this game, i.e., a Nash equilibrium (σ_1, σ_2) with $\sigma_i(s_i) > 0$ for each s_i and each i.

(c) Are there any other NE in this game? [Hard!]

Bayes-Nash equilibrium. Daenerys Targaryen (player 1) and Jon Snow (player 2) are deciding whether or not to form an alliance. Each player $i \in \{1, 2\}$ has an army (including a few dragons) whose value to i is $\theta_i \in [0, 1]$. The process of (possibly) forming an alliance goes as follows:

> Daenerys and Jon simultaneously announce "Yes" (Y) or "No" (N). If both of them announce "Yes", then an alliance is formed. If at least of them announces "No", then there is no alliance, and each of them goes their own way.

We assume that each player i knows the value of their own army, but does not know the value of the other player's army. The players believe that the values θ_i are drawn independently and identically according to a uniform distribution $\mathsf{Unif}[0, 1]$. Daenerys and Jon's payoffs depend on their values and their actions, and are as follows:

$$\pi_1(a_1, a_2, \theta_1, \theta_2) = \begin{cases} \dfrac{\theta_1^2 + 5\theta_2^2}{2} & \text{if } a_1 = a_2 = Y; \\ \theta_1 & \text{otherwise,} \end{cases}$$

$$\pi_2(a_1, a_2, \theta_1, \theta_2) = \begin{cases} \dfrac{5\theta_1^2 + \theta_2^2}{2} & \text{if } a_1 = a_2 = Y; \\ \theta_2 & \text{otherwise.} \end{cases}$$

Observe that if an alliance is formed, each player loses some value of their own army (since $x^2/2 \leq x$) but in return gains some value from the other player's army. Intuitively, we expect a player would hesitate to form an alliance if their value for their own army is high.

A pure strategy s_i of player i specifies an action $s_i(\theta_i) \in \{Y, N\}$ for each possible value $\theta_i \in [0, 1]$. Your goal is to find a *symmetric* pure Bayes-Nash equilibrium (BNE) for this game, i.e., a BNE (s_1, s_2) for which $s_1(x) = s_2(x)$ for all $x \in [0, 1]$.

(a) Suppose Jon follows a strategy s_2 where he announces "Yes" if and only if his value θ_2 is less than (or equal to) some value $x_2 \in (0, 1)$:

$$s_2(\theta_2) = \begin{cases} Y & \text{if } \theta_2 \leq x_2; \\ N & \text{otherwise.} \end{cases}$$

Compute Daenerys's *interim* expected payoff $\Phi_1(a_1 | \theta_1, s_2)$ for each action $a_1 \in \{Y, N\}$ and all $\theta_1 \in [0, 1]$.

You may use the following fact about uniformly distributed random variables:

Fact: If X is $\mathsf{Unif}[0, 1]$, then $\mathbf{E}[X^2 | X \leq a] = a^2/3$ for all $a \in (0, 1)$.

(b) What inequalities must Φ_1 satisfy, for Daenerys's strategy s_1 to be a best response against s_2?

(c) Assuming Jon follows the strategy s_2 in part (b), compute Daenerys's best response strategy s_1^*. Show that in the best response, Daenerys will announce "Yes" if the value of her army is less than (or equal to) some $x_1^* \in [0, 1]$. Find x_1^* in terms of x_2.

(d) Find the value of $x \in (0, 1)$ for which the strategy profile where both Daenerys and Jon announce "Yes" if and only if $\theta_i \leq x$ is a symmetric, pure Bayes-Nash equilibrium.

Infinitely repeated prisoner's dilemma. Recall the prisoner's dilemma G:

	S	B
S	$-1, -1$	$-10, 0$
B	$0, -10$	$-5, -5$

The two players play G repeatedly at times $t = 0, 1, 2, \ldots$. The payoff of player i is given by

$$\pi_i(s_i, s_{-i}) = \sum_{t=0}^{\infty} \delta^t \pi_i^G(a_i^t, a_{-i}^t).$$

Consider the following modification of the grim-trigger strategy for player 1. Player 1 starts by playing S (stay silent) and continues doing so until player 2 first plays B (betray). Thereafter, player 1 continues playing B, until the first time player 2 plays S, at which point player 1 forgives player 2 and starts playing S (until the next betrayal by player 2 when the same cycle of punishment and forgiveness continues). We will call such a player 1 a *forgiver*. We are interested in the case where both players are forgivers and ask when such strategies form an SPNE.

In order to define the strategy formally, we need to define a (common) state of the game X_t at time t to denote whether the game is in cooperation mode ($X_t = 0$), or whether player 1 is being punished ($X_t = 1$) or whether player 2 is being punished ($X_t = 2$). The state evolves as follows: First, the game starts in cooperation mode: $X_0 = 0$. For any $t \geq 1$, we have

$$X_{t+1} = \begin{cases} 0, & \text{if } X_t = 0, \text{ and } (S,S) \text{ or } (B,B) \text{ was played at time } t; \\ 1, & \text{if } X_t = 0, \text{ and } (B,S) \text{ was played at time } t; \\ 2, & \text{if } X_t = 0, \text{ and } (S,B) \text{ was played at time } t; \\ & \\ 0, & \text{if } X_t = 1, \text{ and } (S,B) \text{ was played at time } t; \\ 1, & \text{if } X_t = 1, \text{ and } (S,B) \text{ was } \textbf{not played} \text{ at time } t; \\ & \\ 0, & \text{if } X_t = 2, \text{ and } (B,S) \text{ was played at time } t; \\ 2, & \text{if } X_t = 2, \text{ and } (B,S) \text{ was } \textbf{not played} \text{ at time } t. \end{cases}$$

Define $\mathsf{forgiver}_1$ to be the following strategy for player 1: play S if $X_t = 0$ or $X_t = 1$, and play B if $X_t = 2$. Similarly, define $\mathsf{forgiver}_2$ to be the following strategy for player 2: play S if $X_t = 0$ or $X_t = 2$, and play B if $X_t = 1$.

(a) State the path of play (the sequence of action profiles at each time) under the strategy profile $(\mathsf{forgiver}_1, \mathsf{forgiver}_2)$. What is player 1's expected payoff under this strategy profile? (Your answer will be in terms of δ.)

(b) For each value of $x \in \{0, 1, 2\}$, suppose the state at time t is given by $X_t = x$. What is the path of play under the one-step deviation for player 1? What is player 1's expected payoff under this deviation in this subgame?

(c) For what values of $\delta \in (0, 1)$ is the strategy profile $(\mathsf{forgiver}_1, \mathsf{forgiver}_2)$ an SPNE of $G(\infty, \delta)$? (Be sure to state the *principle* you are using to conclude the result.)